Étale Cohomology of Curves

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Abstract

The main goal of this talk will be to calculate the cohomology of the étale sheaf μ_n over a connected nonsingular curve over an algebraically closed field k, where n is prime to the characteristic of k. We will follow chapters 13 and 14 in [1].

1 Introduction

This talk will be an exercise in computation. The goal is to calculate the cohomology of the étale sheaf μ_n over a connected nonsingular curve X over an algebraically closed field k. Recall that, over $\mathbb{Z}$, the sheaf μ_n is represented by $\operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$. Over a general scheme X, the sheaf μ_n sends an étale open U of X to the nth roots of unity of $\Gamma(U, \mathcal{O}_U)$. Our strategy is as follows:

1. use the Weil Divisor Sequence to calculate the cohomology of $\mathbb{G}_m$, then
2. use the Kummer Sequence to connect the cohomology of $\mathbb{G}_m$ to the cohomology of μ_n.

We will need results from the theory of quasi-algebraically closed fields, as well as the theory of Jacobian varieties, and we will state them without proof as needed.

2 Exactness of the Weil Divisor Sequence and Cohomology of $\mathbb{G}_m$

Using Čech cohomology, we saw that $H^1(X_{\text{ét}}, \mathbb{G}_m) \cong H^1(X_{\text{zar}}, \mathcal{O}_X^\times) \cong \operatorname{Pic}(X)$. We will now compute the rest of the cohomology of $\mathbb{G}_m$ in the case that X is a connected nonsingular variety over an algebraically closed field. In particular, we wish to show that it vanish in the étale setting, just as it does in the Zariski setting. Our computation of the cohomology $H^r(X_{\text{ét}}, \mathbb{G}_m)$ will be based somewhat on the analogous calculation in the Zariski setting. Let us briefly review how thar argument goes.

2.1 The Zariski Setting

Let X be a connected nonsingular variety. The calculation of $H^r(X_{\text{zar}}, \mathcal{O}_X^\times)$ begins by showing that the sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow K^\times \longrightarrow \operatorname{Div}_X \longrightarrow 0$$

is exact. From here, one uses the long exact sequence associated to this short exact sequence and the fact that $K^\times$ and Div_X are flasque sheaves to conclude that $H^r(X_{\text{zar}}, \mathcal{O}_X^\times) = 0$ for $r > 1$.

2.2 The Weil Divisor Sequence in the Étale Setting

The calculation of $H^r(X_{\text{ét}}, \mathbb{G}_m)$ begins with an analogous result. We simply need to replace the Zariski sheaves with the corresponding étale sheaves. In particular, $\mathcal{O}_X^\times$ is replaced by $\mathbb{G}_m$, $K^\times$ is replaced by $g_*\mathbb{G}_{m,K}$ (where $g : \eta \to X$ is the generic point), and Div_X is replaced by

$$\text{Div}_X = \bigoplus_{\text{codim}(z)=1} i_{z,*}\mathbb{Z}.$$

Proposition 2.2.1. *Let X be a connected nonsingular variety with generic point $g : \eta \to X$. The sequence étale sheaves of abelian groups*

$$0 \longrightarrow \mathbb{G}_m \longrightarrow g_*\mathbb{G}_{m,\eta} \longrightarrow \text{Div}_X \longrightarrow 0$$

is exact.

Proof. It suffices to check that for any étale U over X, the restriction of this sequence to U_{zar} is exact. However, the resulting sequence is the usual Weil Divisor Sequence, which is known to be exact. □

We still have the long exact sequence induced by short exact sequences. Unfortunately, we no longer have a notion of flasqueness. We will thus need to be creative to conclude that $g_*\mathbb{G}_{m,\eta}$ and Div_X have no higher cohomology. We will need some machinery to get through this.

2.3 Results on Quasi-Algebraically Closed Fields

Definition 2.3.1. A field k is said to be quasi-algebraically closed if every nonconstant homogeneous polynomial f in n variables with $\deg(f) < n$ has a zero.

Example 2.3.2. We have the following examples of quasi-algebraically closed fields.

1. Finite fields are quasi-algebraically closed.
2. (Tsen's Theorem) Let k be an algebraically closed field and let K be a finitely generated extension of K of transcendence degree 1. Then K is quasi-algebraically closed.
3. (Lang) Let R be a Henselian discrete valuation ring with algebraically closed residue field, and let K be the field of fractions of R. If the completion of K is seperable over K, then K is quasi-algebraically closed.

The notion of quasi-algebraically closed fields becomes relevant to us in light of the following proposition.

Proposition 2.3.3. *Let k be a quasi-algebraically closed field and let $G = \text{Gal}(k^{sep}/k)$. Then*

1. *$H^2(G, (k^{sep})^\times) = 0$,*
2. *for any $r > 1$, $H^r(G, M) = 0$ if M is a torsion discrete G-module, and*
3. *for any $r > 2$, $H^r(G, M) = 0$ if M is a discrete G-module.*

The proofs of these propositons are somewhat involved, so we will taken them as granted.

2.4 Computation of $H^r(X_{\text{ét}}, \mathbb{G}_m)$

Proposition 2.4.1. *Let X be a connected nonsingular curve over an algebraically closed field k. Then*

$$H^r(X_{\acute{e}t}, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^\times) & r = 0 \\ \text{Pic}(X) & r = 1 \\ 0 & r > 1 \end{cases}.$$

Proof. We have already computed H^r for $r = 0$ and $r = 1$ using Čech cohomology. It remains show that $H^r(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $r > 1$. From the Weil Divisor sequence, we get the following long exact sequence

$$\cdots \longrightarrow H^{r-1}(X_{\text{ét}}, \text{Div}_X) \longrightarrow H^r(X_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta}) \longrightarrow H^r(X_{\text{ét}}, \text{Div}_X) \longrightarrow \cdots$$

where $g : \eta \to X$ is the generic point. It thus suffices to show that $H^r(X_{\text{ét}}, \text{Div}_X)$ vanishes for $r > 0$, and that $H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta})$ vanishes for $r > 1$.

Note that for any closed point $i_z : z \to X$, the functor $i_{z,*} : \mathbf{Sh}(z_{\text{ét}}) \to \mathbf{Sh}(X_{\text{ét}})$ is exact. The Leray Spectral Sequence then tells us that $H^r(X_{\text{ét}}, i_{z,*}\mathbb{Z}) = H^r(z_{\text{ét}}, \mathbb{Z})$ for all r. Since we are working over an algebraically closed field k, we have that $H^r(z_{\text{ét}}, \mathbb{Z}) = 0$ for $r > 0$. Recalling that

$$\text{Div}_X = \bigoplus_{\text{codim}(z)=1} i_{z,*}\mathbb{Z} = \bigoplus_{\text{closed points } z \in X} i_{z,*}\mathbb{Z}$$

we conclude that $H^r(X_{\text{ét}}, \text{Div}_X) = 0$ for $r > 0$.

The computation of $H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta})$ will also make use of the Leray Spectral Sequence. A result (Example 12.4 of [1]) shows that

$$(\mathbf{R}^r g_*\mathbb{G}_{m,\eta})_{\overline{y}} = \begin{cases} 0 & \text{if } y = \eta \text{ and } r > 0 \\ H^r(\text{Spec}\, K_{\overline{x}}, \mathbb{G}_m) & \text{if } y = x \neq \eta \end{cases}$$

Here $K_{\overline{x}}$ is the field of fractions of the Henselian discrete valuation ring $\mathcal{O}_{X,x}$. The results on quasi-algebraically closed fields, together with Hilbert's Theorem 90, imply that $H^r(\text{Spec}\, K_{\overline{x}}, \mathbb{G}_m)$ vanishes for $r > 0$. Thus, $\mathbf{R}^r g_*\mathbb{G}_{m,\eta} = 0$ for $r > 0$, and so the Leray Spectral Sequence tells us that

$$H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta}) = H^r(G, (K^{\text{sep}})^\times)$$

for all r, where K is the field of fractions of X and $G = \text{Gal}(K^{\text{sep}}/K)$. Again, by the results on quasi-algebraically closed fields and Hilbert's Theorem 90, we have that $H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,\eta}) = 0$ for $r > 0$. □

3 Exactness of the Kummer Sequence and Cohomology of μ_n

Our next goal is to calculate $H^r(X_{\text{ét}}, \mu_n)$ for X a connected nonsingular curve over an algebraically closed field k and n prime to the characteristic of k. As in the calculation of the cohomology of $\mathbb{G}_m$, this calculation will be made possible by a short exact sequence of étale sheaves.

3.1 The Kummer Sequence

Proposition 3.1.1. *Let X be a variety over a field k, and let n be a number prime to the characteristic of k. Then the sequence of étale sheaves of abelian groups*

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 0$$

is exact.

Proof. We will check exactness at the level of stalks. Recall that the geometric stalks of the structure sheaf are strictly Henselian rings (i.e. Henselian with separably closed residue field). Thus, exactness of this sequence at the level of stalks is equivalent to exactness of

$$0 \longrightarrow \mu_n(A) \longrightarrow A^\times \xrightarrow{n} A^\times \longrightarrow 0$$

for every strictly Henselian ring A. The only nontrivial part is showing that every element of A is an nth power.

Let $a \in A^\times$ and let $\overline{a}$ be its reduction to the residue field of A. Since the residue field is separably closed, $x^n - \overline{a}$ has a solution, provided n is prime with the characteristic of k. Since A is Henselian, this solution lifts to a solution of $x^n - a$ in A. □

3.2 Computation of $H^r(X_{\text{ét}}, \mu_n)$

First, we will consider the case that X is a complete.

Proposition 3.2.1. *Let X be a complete connected nonsingular curve of genus g over an algebraically closed field k. For any number n prime to the characteristic of k, we have*

$$H^r(X_{\acute{e}t}, \mu_n) \cong \begin{cases} \mu_n(k) & r = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & r = 1 \\ \mathbb{Z}/n\mathbb{Z} & r = 2 \\ 0 & \textit{else} \end{cases}.$$

Proof. The long exact sequence assoicated to the Kummer sequence is as follows.

$$\cdots \longrightarrow H^{r-1}(X_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^r(X_{\text{ét}}, \mu_n) \longrightarrow H^r(X_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^r(X_{\text{ét}}, \mathbb{G}_m) \longrightarrow \cdots$$

This immediately implies the result for $r > 2$. For $r = 1$, notice that the map $H^0(X, \mathbb{G}_m) \to H^0(X, \mathbb{G}_m)$ is given by multiplication by n on $k^\times$. Since k is algebraically closed, this map is surject and thus the long exact sequence into the following.

$$0 \longrightarrow H^1(X_{\text{ét}}, \mu_n) \longrightarrow \text{Pic}(X) \xrightarrow{\ n\ } \text{Pic}(X) \longrightarrow H^2(X_{\text{ét}}, \mu_n) \longrightarrow 0$$

Now, a result from the theory of Jacobian varieties tells us that the map $n : \text{Pic}(X) \to \text{Pic}(X)$ has kernel isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ and cokernel isomorphic to $\mathbb{Z}/n\mathbb{Z}$ when n is prime to the characteristic. □

To get the not complete case, it suffices to consider the cohomology of a curve relative to a closed point. We can then recover the cohomology of a curve that is not necessarily complete via the long exact sequence of a pair of spaces.

Proposition 3.2.2. *Let X be a connected nonsingular curve over an algebraically closed field k, and let $x \in X$ be a closed point. If n is prime to the characteristic of k, then*

$$H^r_x(X_{\acute{e}t}, \mu_n) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & r = 2 \\ 0 & \textit{else} \end{cases}.$$

Proof. Let R be the strictly local ring of x. We can then use excision to replace X with $\text{Spec}\, R = V$. As in the absolute case, we will first compute the relative cohomology of $\mathbb{G}_m$ and then calculate the relative cohomology of μ_n.

By the computation of cohomology of $\mathbb{G}_m$, we have that $H^r(V_{\text{ét}}, \mathbb{G}_m) = 0$ for $r > 1$. Moreover, we have $H^1(V_{\text{ét}}, \mathbb{G}_m) = 0$ since R is a discrete valuation ring. The long exact sequence of the pair $(V, V \setminus x)$ is given by

$$\cdots \longrightarrow H^{r-1}(\text{Spec}\, K, \mathbb{G}_m) \longrightarrow H^r_x(V_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^r(V_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^r(\text{Spec}\, K, \mathbb{G}_m) \longrightarrow \cdots$$

where K is the field of fractions of V. (Note that $\text{Spec}\, K = V \setminus x$.) Thus, for $r > 1$, we have an isomorphism $H^{r-1}(\text{Spec}\, K, \mathbb{G}_m) \cong H^r_x(V_{\text{ét}}, \mathbb{G}_m)$. Using the results on quasi-algebraically closed fields and Hilbert's Theorem 90, we have that $H^r(\text{Spec}\, K, \mathbb{G}_m)$ vanishes for $r > 0$, so $H^r_x(V_{\text{ét}}, \mathbb{G}_m)$ vanishes for $r > 1$. The fact that $H^0_x(V_{\text{ét}}, \mathbb{G}_m) = 0$ is obvious.

Finally, $H^1(V_{\text{ét}}, \mathbb{G}_m) \cong H^0(\text{Spec}\, K, \mathbb{G}_m)/H^0(V, \mathbb{G}_m) \cong \mathbb{Z}$. The result now comes immediately from the Kummer Sequence. □

References

[1] James S. Milne. Lectures on etale cohomology (v2.10), 2008. Available at www.jmilne.org/math/.