Questions from last time

Remark 1. A question that came up last time: What's the difference between μ_n and $\mathbb{Z}/n$? First, recall that from Brian's talk, we know that over Spec$\mathbb{Z}$, we have μ_n is the sheaf represented by $\mathbb{Z}[x]/(x^n - 1)$. Now, given an arbitrary scheme X, we have a canonical map $f : X \to \operatorname{Spec}\mathbb{Z}$. And hence a morphism $f^* : Sh(\operatorname{Spec}\mathbb{Z}_{et}) \to Sh(X_{et})$. The sheaf μ_n is $f^*(\mu_n^{\mathbb{Z}}) = Hom(-, \operatorname{Spec}\mathbb{Z}[x]/(x^n-1) \times_{\operatorname{Spec}\mathbb{Z}} X$. Thus we have that $Hom(\mu_n^X, P) = Hom(\mu_n^{\mathbb{Z}}, f_*P) = f_*(P)(\mu_n^{\mathbb{Z}}) = P(\operatorname{Spec}\mathbb{Z}[x]/(x^n-1) \times_{\operatorname{Spec}\mathbb{Z}} X)$.

First, say k is a field with all nth roots of unity. Then on $\operatorname{Spec} k_{et}$, I claim that $\mu_n \cong \mathbb{Z}/n$. The sheaves are clearly the same on all sections, the only thing to worry about might be that the isomorphism $\mu_n \cong \mathbb{Z}/n$ isn't natural. However, by definition, $\mu_n = Hom_{\operatorname{Spec} k_{et}}(-, \operatorname{Spec} k[T]/(T^n-1)) \cong Hom_{Sepkalg}(k[T]/(T^n - 1), \gamma(-))$. Etale schemes over $\operatorname{Spec} k$ come with a map to $X \to \operatorname{Spec} k$, which means they naturally come with an injection $k \to \Gamma(X)$. Thus we just need to choose a primitive root of unity ξ in $\operatorname{Spec} k$, and we get a natural identification of $Hom_{Sepkalg}(X, \operatorname{Spec} k[T]/T^n - 1) \cong \mathbb{Z}/n$ sending 1 in $\mathbb{Z}/n$ to the map $Hom_{Sepkalg}(k[T]/(T^n-1), \gamma(X))$ which sends T to ξ.

First, let me make some remarks about terminology. Throughout the course, we've been letting $j_!$ denote extension by zero, which we viewed as the left adjoint of j^*. However, recall that we only defined $j_!$ for open immersions. Let me remark briefly about what's going on, as this will come up when we start talking about Poincaré duality.

1 Recollection of Topology

Definition 2. Let $f : X \to Y$ be a continuous between locally compact spaces. For a sheaf F on X and an open subset V of Y, put

$$\Gamma(V, f_!F) = \{s \in \Gamma(f^{-1}(V), F) \mid Supp(s) \xrightarrow{f} Y \text{ is a proper map } \}$$

This definition has three crucial properties

1. If $j : U \to Y$ is an open immersion, then $j_! = j_\#$ is extension by zero. Let's check at least that the stalks outside of $j(U)$ are zero. Fix an $x \notin j(U)$. Then fix an open set V containing x so that $V \cap j(U)^C \neq \emptyset$. Now fix some section $s \in j_!(V)$. By definition, $j : Supp(s) \subset U \to Y$ is a proper map. Furthermore, since the spaces are locally compact, we know that there exists a compact set $K \subset V$ such that there exists an open neighborhood W of x contained in K. Now $K \cap Supp(s)$ is compact by assumption, so $T = W - (K \cap Supp(s))$ is an open neighborhood of x s.t. $s \mid_f^{-1} (T) = 0$ just as desired.
2. If $p : X \to Y$ is a proper map, then $p_! \cong p_*$. For this, fix any open subset V of Y. Then for any $s \in F(f^{-1}(V))$, the map $Supp(s) \hookrightarrow X \to Y$ is proper, since closed subspaces of compact spaces are compact.
3. If $f = p \circ j$ where p is proper and j is an open immersion, then $f_! = p_! \circ j_! = p_* \circ j_\#$. This is just saying that $(-)_!$ is a functor and using the previous two results.

Definition 3. Define the compact global sections functor as $\Gamma_c(X; F) = f_!F$, where $f : X \to *$. The right derived functors of this are cohomology with compact support.

2 Pushforward with Compact Support for Étale Sheaves

Recall that j^* always has a right adjoint j_*.

Definition 4. Say $j : Y \to X$ is an étale map so that j^* is just restriction to Y_{et}. Then j^* has a left adjoint $j_\#$ called *extension by zero.*

To define pushforward with compact support, we need a theorem.

Theorem 1. *(Nagata's Compactification Theorem)*

A separated and finite type morphism to a qcqs scheme S can be factored into an open immersion followed by a proper mapping.

Remark 5. To the best of my knowledge, this is only functorial in dimension 1. We shouldn't expect functoriality in general, as classically cohomology with compact support is only functorial for *proper* maps.

Remark 6. Let's recall precisely what we mean by functoriality of étale cohomology. There are really two different types of functoriality going here.

1. If $f : F \to G$ are both sheaves on X_{et}, then the usual lemmas in abelian categories which give rise to a map of injective resolutions give us a map $H^i(X_{et}, F) \to H^i(X_{et}, G)$.

2. If we have a map of schemes $\phi : X \to Y$, we get a morphism of topoi $\phi^* : Sh(Y_{et}) \to Sh(X_{et})$. Let $\mathbb{Z}_Y$ denote the constant presheaf on Y_{et} and

$$a : PSh(Y_{et}) \rightleftarrows Sh(Y_{et}) : i$$

the sheafification adjoint pair. Fix a presheaf B. It's trivial to see that $Hom_{Ab}(\mathbb{Z}, \Gamma(B)) \cong Hom_{Psh}(\mathbb{Z}_Y, B)$. Now if B is actually a sheaf, then we have $Hom_{Psh}(\mathbb{Z}_Y, B) \cong Hom_{Psh}(\mathbb{Z}_Y, i(B)) \cong Hom_{Sh}(a(\mathbb{Z}_Y), B)$ so that $Hom_{Ab}(\mathbb{Z}, \Gamma(B)) \cong Hom_{Sh}(a(\mathbb{Z}_Y), B)$ and we have that global sections is the right adjont of the constant sheaf functor.

I claim now that $\phi^*(a(\mathbb{Z}_Y)) \cong a(\mathbb{Z}_X)$. We compute:

$$Hom_{Sh(X_{et})}(\phi^*(a(\mathbb{Z}_Y)), B) \cong Hom_{Sh(Y_{et})}(a(\mathbb{Z}_y), \pi_*(B)) \cong Hom_{Ab}(\mathbb{Z}, \Gamma(\pi_* B)).$$

But $\Gamma(\pi_* B) \cong B(X \times_X Y) = B(Y)$. So the above is $Hom_{Ab}(\mathbb{Z}, \Gamma(B))$ which is naturally isomorphic to $Hom_{Sh(X_{et})}(a(\mathbb{Z}_X), B)$. Hence, by uniqueness of adjoints, $\phi^*(a(\mathbb{Z}_Y)) \cong a(\mathbb{Z}_X)$.

It follows that the functor ϕ^* induces a natural map $H^0(Y_{et}, F) \to H^0(X_{et}, \phi^* F)$. Using that ϕ^* is exact, it follows that $F \mapsto H^i(X_{et}, \phi^* F)$ is a cohomological δ functor. Now by universality of right derived functors, it follows that there are maps $H^i(Y_{et}, F) \to H^i(X_{et}, \phi^* F)$ for all i.

Now, we can define the direct image functors:

Definition 7. Let f be a morphism as in the theorem, and factor $f = p \circ j$ into an open immersion followed by a proper map. Define $f_! = p_* \circ j_\#$.

Definition 8. Fix a scheme U, and let f be the map $f : U \to \operatorname{Spec} \mathbb{Z}$ to the terminal object. Nagata's theorem applied to this map produces a proper scheme $\overline{U}$ and an open immersion $j : U \to \overline{U}$. Define $H_c^R(U, \mathcal{F}) = H^r(\overline{X}, j_! F)$.

Remark 9. If we have a scheme or variety over some field, we replace the map to $\operatorname{Spec} \mathbb{Z}$ with the map to spec of that field.

Remark 10. There are a lot of potential issues with the definition.

- One is that *a priori* the definition depends on a choice of factorization of f. As we remarked above, for curves there's a functorial such choice. In general there isn't, but one can show that for torsion sheaves F the cohomology is independent of the factorization.

- Another is that $j_!$ doesn't preserve injectives, so that $H_c^r(U, F)$ is not the rth right derived functor of $H_c^0(U, -)$. Indeed, if it was, we would compute $H_c^r(U, F)$ by taking an injective resolution of $\mathcal{F}$, then applying $\Gamma(\overline{U}, j_!(-))$. But this does NOT yield an injective resolution of $j_!(F)$.

- As remarked above, except in dimension one, this factorization is not functorial. In dimension one, we use that every function field K in one variable has a connected complete regular curve X canonically assocaited with it.

- Since $j_!$ is exact, we DO get a long exact sequence in compactly supported cohomology.

Theorem 2. *For any connected regular curve U over an algebraically closed field k and integer n not divisible by the characteristic of k, there is a canonical isomorphism*

$$H_c^2(U, \mu_n) \to \mathbb{Z}/n\mathbb{Z}.$$

Proof. Let $j : U \hookrightarrow X$ be the canonical inclusion of U into a complete regular curve, and let $i : Z \hookrightarrow X$ be the complement of U in X. Regard μ_n on a sheaf on X.

Lemma 11. *There is an exact sequence*

$$0 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 0.$$

Proof. It suffices to check exactness on stalks. If $\overline{x}$ has image in the complement of U, then by definition $(j_! j^* \mu_n)_{\overline{x}} = 0$, and the stalk $(i_* i^* \mu_n)_{\overline{x}}$ is the stalk $(i^* \mu_n)_{\overline{x}}$. Recall that the stalks of $i^* \mu_n$ are the stalks of μ_n at the geometric point $\overline{x} \to Z \xrightarrow{i} X$. Thus in this case we just recover $(\mu_n)_{\overline{x}}$, and hence the sequence is exact.

If $\overline{x}$ has image in U, then the first two terms agree, and the last is zero. □

The above sequence yields a LES

$$\cdots \longrightarrow H^r_c(U, \mu_n) \longrightarrow H^r(X, \mu_n) \longrightarrow H^r(X, i_* i^* \mu_n) \longrightarrow \cdots$$

Now, the cohomology $H^r(X, i_* i^* \mu_n) \cong H^r(Z, i^* \mu_n)$ by adjunction. Furthermore, $H^r(Z, I^* \mu_n) = 0$ for $r > 0$ by the Kummer sequence, since $\mathbb{G}_m$ is injective over an algebraically closed field (it's a constant sheaf). □

Let's recall what the classical Poincare duality theorem says:

Theorem 3. *Let M be an oriented n-manifold with a finite good cover. There's a nondegenerate pairing*

$$\int_M : H^q(M) \otimes H^{n-q}_c(M) \to \mathbb{R}.$$

The proof is a standard five-lemma/Mayer-Vietoris argument and induction on the cardinality of a good cover.

Theorem 4. *For any finite locally constant sheaf $\mathcal{F}$ on U and integer $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times H^{2-r}(U, \check{F}(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

A pairing $M \times N \to C$ is said to be perfect *if the induced maps*

$$M \to Hom(N, C), \qquad N \to Hom(M, C)$$

are isomorphisms. The sheaf $\check{F}(1)$ is

$$V \mapsto Hom_V(\mathcal{F} \mid_V, \mu_n \mid V).$$

We'll need to black box some results as we discuss the proof of this theorem. First, for a variety X over an algebraically closed field k,

$$cd(X) \leq 2 \dim(X).$$

Thus for a complete curve X over an algebraically closed field, we have $cd(X) \leq 2$, so that $H^i(X_{et}, \mathcal{F}) = 0$ for $i > 2$ and all torsion sheaves $\mathcal{F}$ on X_{et}. Thus the proof of the theorem is dedicated to proving it for $r = 0, 1, 2$. We'll omit the proof.

Here's a generalization which will allow us to actually define the pairing in Poincare duality:

Theorem 5. *Let U be a nonsingular curve over and algebraically closed field k. For all constructible sheaves $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$ modules on U and all $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times Ext^{2-r}_{U,n}(\mathcal{F}, \mu_n) \to H^r_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Definition 12. A sheaf $\mathcal{F}$ on a curve U over a field k is *constructible* if

1. $\mathcal{F} \mid V$ is locally constant for some nonempty open subset V of U;

2. the stalks of $\mathcal{F}$ are finite.

Definition 13. Let $Sh(X_{et}, n)$ be the category of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et}. Then $Ext^r(F, G)$ is $R^r Hom(F, G)$ where the Hom is computed in $Sh(X_{et}, n)$.

Lemma 14. *If F is locally constant, then $Ext^r_{U,n}(F, \mu_n) \cong H^r(U_{et}, \check{F}(1))$, where $\check{F}(1)$ is the sheaf*

$$V \mapsto Hom_V(F\mid_V, \mu_n).$$

Proof. Let $\underline{Ext}^s(F, \mu_n)$ denote the right derived functors of sheaf hom. There's a spectral sequence

$$H^r(Y_{et}, \underline{Ext}^s(F, \mu_n)) \implies Ext^s(F, \mu_n).$$

On stalks, we have

$$\underline{Ext}^r(F, \mu_n)_{\overline{y}} = Ext^r(F_{\overline{y}}, (\mu_n)_{\overline{y}}).$$

where the Ext on the right is in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. Now $\mathbb{Z}/n\mathbb{Z}$ is injective as a module over itself as can be seen via Baer's criterion: given a morphism $m\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, m must map to an element k which it divides mod n. We extend the map to $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by sending 1 to k/m, and hence Baer's criterion is satisfied. so these groups are 0 for $r > 0$. Thus the spectral sequence collapses to yield

$$H^r(X_{et}, \underline{Hom}(F, \mu_n)) \cong Ext^r_{X,n}(F, \mu_n)$$

□

Hence for any locally constant sheaf F, the pairing above takes the form

$$H^r_c(U, F) \times H^{2-r}(U, \check{F}(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

Now, to get a pairing

$$Ext^s(F, \mu_n) \times H^r(X, j_! F) \to H^{r+s}(X, j_1 \mu_n)$$

we represent an element of $Ext^s(F, \mu_n)$ as an exact sequence

$$0 \to \mu_n \to E_1 \to \cdots \to \cdots \to E_s \to F \to 0.$$

Now apply $j_!$ to get an extension

$$0 \to j_! \mu_n \to j_! E_1 \to \cdots \to j_! E_s \to j_! F \to 0$$

break this up into short exact sequences

$$0 \to K_i \to j_! E_i \to K_{i+1} \to 0$$

and now form the iterated boundary map (coming from the snake lemma)

$$H^r(X, j_! F) \to H^{r+1}(X, K_s) \to \cdots \to H^{r+s}(X, j_! \mu_n)$$

to define a pairing.

Remark 15. The theorem above is similar to the classical theorem in that we already know how to prove it for locally constant sheaves, so now we use a 5-lemma argument and the fact that constructible sheaves are locally locally constant.

Unfortunately, most of the applications of Poincaré duality require us to define the cycle class map. For curves, the cycle class map is just the map $Pic(X) \to H^2(X_{et}, \mathbb{Z}/n\mathbb{Z})$ defined by the LES associated to the Kummer sequence.