This is a generalization of the story I discussed in my Geom/Top talk.

Definitions and useful results

Definition 1. (The obvious one) A morphism $Y \to X$ is finite étale if it is finite and étale.

I need a preliminary definition before I can give a more useful description of finite étale maps.
Definition 2. Let B be a finitely generated projective A-module. We say that B is separable over A if the map $\phi : B \to Hom_A(B, A)$, $\phi(x)(y) = Tr_{B/A}(xy)$ is an isomorphism. Here the trace is the composite

$$End_A(B) \xrightarrow{\psi^{-1}} B^* \otimes_A B \to A$$

where the first map is the inverse of the isomorphism $\psi : B^* \otimes_A B \to End_A(B)$ given by $\psi(f \otimes b)(p) = f(p) \cdot b$. The second map is the usual evaluation map.
Definition 3. (The useful one) A morphism $f : Y \to X$ is finite étale if there exists an affine open cover $U_i = \operatorname{Spec} A_i$ of X such that each $f^{-1}(U_i)$ is affine and equal to $\operatorname{Spec} B_i$, where B_i is a free separable A_i-algebra. Equivalently, we could require that for EVERY open affine subset $U_i = \operatorname{Spec} A_i$ of X, the inverse image is the spectrum of a finite projective separable A_i-algebra.
Remark 4. These two definitions agree if X is locally noetherian.

Now, fix a connected, locally noetherian scheme X, and let FEt/X denote the category of finite étale maps over X. Even if we don't require it, a map $B \to X$ to $B' \to X$ will necessarily be a finite étale map $B \to B'$ making the obvious triangle commute (compare this to the fact for covering spaces).

Here's an essential result that is the analogue of "local triviality of covering spaces" for finite étale morphisms.
Definition 5. A morphism $f : Y \to X$ of schemes is *totally split* if X can be written as a disjoint union of schemes $X_n, n \in \mathbb{Z}$, $n \geq 0$, s.t. for each n the scheme $f^{-1}(X_n)$ is isomorphic to the disjoint union of n copies of X_n with the natural morphism $X_n \coprod \cdots \coprod X_n \to X_n$.
Theorem 1. *Let $f : Y \to X$ be a morphism of schemes. Then f is finite étale if and only if f is affine and $Y \times_X W \to W$ is totally split for some $W \to X$ that is surjective, finite, and locally free.*

Proof. Idea: induct on the degree of the finite étale morphism. □

Construction of the étale fundamental group

Fix a connected base scheme X, and a geometric point $\overline{x} : \operatorname{Spec} \Omega \to X$, where Ω is a separably closed field. For simplicity, we'll assume that X is locally noetherian, that is, it has a cover by affine schemes, each of which is the spectrum of a Noetherian ring. This simplification will allow me to not talk about "finitely presented" morphisms instead of finite morphisms.

Idea: The étale fundamental group is "the thing which classifies finite étale coverings. In other words, once we define $\pi_1^{et}(X, x)$, there will be an equivalence of categories from $FEt/X \to \pi_1^{et}(X, x)$-sets.
Definition 6. Let $F_x : FEt/X \to$Sets be the functor which assigns to an étale covering $Y \to X$ the fiber $Y \times_X \operatorname{Spec} \Omega$.

To expand on this a bit, we know that finite étale morphisms are stable under base change, so $Y \times_X \operatorname{Spec} \Omega \to \operatorname{Spec} \Omega$ is a finite étale map to the spectrum of an algebraically closed field. Thus the fiber is necessarily just a finite disjoint union of points, as we saw last time a description of finite etale maps to spectra of fields.
Definition 7. The étale fundamental group $\pi^{et}(X, x)$ is defined to be $\operatorname{Aut}(F_x)$, the group of natural isomorphisms from F_x to itself. We can view this as a subgroup of $\prod_Y Aut(F_X(Y)) = \prod_Y S_{|F_X(Y)|}$, where S_n is the group of automorphisms of an n-element set. Endowing each S_n with the discrete topology, we give $\pi^{et}(X, x)$ the subspace topology of the product topology.

Here's another construction:
Lemma 8. *Let A be a connected object of FEt/X. Fix $a \in F_x(A)$. Then the map $Hom(A, B) \to F_x(B)$ given by $f \mapsto F_x(f)(a)$ is injective.*

Proof. F_x is a limit, and hence commutes with limits (in particular equalizers). Say $F_x(f)(a) = F_x(g)(a)$. Then in particular $f(a) = g(a)$, so a is in the equalizer of f and g. But equalizers are monomorphisms, so the equalizer is a nontrivial subobject of A, and hence must be all of A so that $f = g$. □

Definition 9. Let $C = FEt/X$. Let A be connected (precisely 2 subobjects). Then by the above lemma, we have $\#Aut_C(A) \leq \#Mor_C(A, A) \leq \#F_x(A)$, since if two finite étale morphisms out of a connected space agree at a point, they agree everywhere. There's an action of $Aut_C(A)$ on $F_x(A)$ given by $f \cdot a = F_x(f)(a)$. We say that A is *Galois* if this action is transitive. In this case, the above inequalities are all equalities.

Denote by J the set of all pairs (A, a), where A is connected and Galois with $a \in F(A)$. Write $(A, a) \geq (B, b)$ if $b = F(f)(a)$ for some $f \in Mor_C(A, B)$. By the above lemma, this map f is unique if it exists, so I is a poset.

Now, let $(A, a), (B, b) \in J$ be such that $(A, a) \geq (B, b)$ with corresponding morphism $f : A \to B$. For each $\sigma \in Aut_C(A)$, there is a unique $\tau \in Aut_C(B)$ for which

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ A & \xrightarrow{f} & B \end{array}$$

commutes, namely the automorphism τ with $F(\tau)(b) = F(f\sigma)(a)$. The map $Aut_C(A) \to Aut_C(B)$ sending σ to τ in this situation is a surjective group homomorphism. Thus we get a projective system and define

$$\pi(X, x) = \varprojlim_J Aut_C(A).$$

Now, we let $\pi(X, x)$ act on the fiber by noting that F is prorepresentable as $F_x(X) \cong \varinjlim_J Hom(A, X)$.
Theorem 2. *The étale fundamental group is the unique up to isomorphism profinite group π such that FEt/X is equivalent to the category of finite π-sets on which π acts continuously.*

Examples

We'll spend the rest of the talk computing some examples.
Example 10. Let $X = \operatorname{Spec} k$, where k is an algebraically closed field. Then $\pi(X, x) = 1$.

Proof. Say η is a natural automorphism of F_x. Consider the diagram

$$\begin{array}{ccc} \{1\} & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \{1, \dots, n\} & \xrightarrow{\sigma} & \{1, \dots, n\} \end{array} \quad .$$

The map on top is necessarily the identity. Furthermore, we can have any map $\{1\} \to \{1, \dots, n\}$. Thus in particular, we'll have $\sigma(m) = m$ for all $m \in \{1, \dots, n\}$. It follows that all the components of the natural transformation must be the identity, and hence $Aut(F_x) = \{1\}$. □

Theorem 3. *Let X be a nonsingular scheme over* $\operatorname{Spec} \mathbb{C}$*, and let X_h denote the associated complex analytic space. Then $\pi^{et}(X) \cong \widehat{\pi}(X_h)$, which is the profinite completion of the fundamental group of X_h.*
Example 11. Let $X = \mathbb{G}_m = \mathbb{A}^1_{\mathbb{C}} - \{0\}$. Then the associated analytic space is $\mathbb{C} - \{0\}$. The fundamental group of this space is $\mathbb{Z}$, so the etale fundamental group of $\mathbb{G}_m$ is $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.
Theorem 4. *Let X be a normal integral scheme, K its function field, $\overline{K}$ an algebraic closure of K, and M the composite of all finite spearable field extensions L of K with $L \subset \overline{K}$ for which X is unramified in L. Then the fundamental group $\pi(X)$ is isomorphic to the Galois group $Gal(M/K)$.*

Recall: If X is a normal integral scheme, and K the function field of X, then for any finite separable field extension L of K we can form the following construction: for an open $U \subset X$, $U \neq \emptyset$, let $A(U)$ be the integral closure of $\mathcal{O}_X(U)$ in L, and $A(\emptyset) = 0$. Then A is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and hence gives rise to an affine morphism $Y \to X$ with $Y = \operatorname{Spec} A$ (form Y locally then glue). This is called the *normalization of X in L*. We'll say that X is unramified in L if $Y \to X$ is unramified.

Remark 12. If X is a locally noetherian of dimension one normal integral scheme, then the M above is the largest extension of K within a fixed separable closure of K in which all valuations induced by the closed points $x \in X$ are unramified.

Proof. (Lenstra-Galois theory for schemes).

□

Corollary 13. Let X be the spectrum of a field K. Then $\pi(X) \cong Gal(K_s/K)$.

This is also clear from the definition of the étale fundamental group above using the limit of the automorphism groups.

Proof. The fields L as above will have all the finite Galois extensions as a cofinal subset. □

Note that this gives another proof that the étale fundamental group of an algebraically closed field is trivial. Note also that for $\operatorname{Spec} \mathbb{Q}$, we get the absolute Galois group of $\mathbb{Q}$.

Example 14. Let $X = \operatorname{Spec} \mathbb{Z}_p$ for some prime number p. Then $K = \mathbb{Q}_p$, and M is the maximal unramified extension of K. Number theorists apparently know that $Gal(M/K) \cong Gal(F_p/\mathbb{F}_p) \cong \widehat{Z}$. Thus $\pi(\operatorname{Spec} \mathbb{Z}_p) \cong \widehat{\mathbb{Z}}$.

Aside: Recall that given an algebraic ring of integers R with fraction field K and a prime P in R, for a fixed extension L of K, we can take the integral closure S of R in L. The extension PS of P to an ideal of S will factor into primes $P_1^{e(1)} \cdots P_k^{e(k)}$ (Dedekind domain). If all these powers are one, the extension is unramified at P.

Definition 15. Let v be a valuation on $K(t)$, F a finite separable field extension, and w and extension of v to F. That is, a valuation $F^* \to \mathbb{Z}$ which restricts to v under the inclusion map. The ramification index of w over v is defined to be $[w(F^*) : v(K(t)^*)]p^i$ where p^i is the inseparable degree of the extension R_w/m_w over R_v/m_v. We say that w is tamely ramified over v if the residue class field extension $\overline{K(t)}_v \subset \overline{F}_w$ is separable and the ramification index $e(w/v)$ is not divisible by $char(K)$.

Theorem 5. *Let K be a field, t transcendental over K, and F a finite separable extension of $K(t)$. Suppose that v_∞ is tamely ramified in F, and that all v_f are unramified in F. Then $F = L(t)$ for some finite separable extension L of K.*

Example 16. ($\mathbb{P}^1_K$).

Let K be a field and let $X = \mathbb{P}^1_K$. The function field of X is $K(t)$, and the closed points of X correspond to valuations ν_f, ν_∞ since the stalks at each point are DVRs. The field M as above will be $K_s(t)$, so that $\pi(X) \cong Gal(K_s(t)/K(t)) \cong Gal(K_s/K) \cong \pi(\operatorname{Spec} K)$. In particular $\pi(X)$ is trivial if K is separably closed.

Example 17. Let K be a field and $X = \mathbb{A}^1_K$. Suppose that $\operatorname{char} K = 0$. Then M is again $K_s(T)$, so that $\pi(\mathbb{A}^1_K) \cong \pi(\operatorname{Spec} K)$.

If $\operatorname{char} K = p > 0$, the natural map $\pi(\mathbb{A}^1_K) \to \pi(\operatorname{Spec} K)$ is still surjective, but it is not injective.

Lemma 18. *Let $\pi' \to \pi$ be a map of profinite groups. Let G be the induced functor $\pi' - sets \to \pi - sets$. This is surjective iff G sends transitive sets to transitive sets, and injective iff for every connected object X' of $\pi' - sets$ there is an object X of $\pi - sets$ and a connected component Y' of $G'(X)$ s.t. there is a π'-homomorphism $Y' \to X'$.*

Example 19. Let A be a finite ring, and suppose that $\operatorname{Spec} A$ is connected. Then A is a local ring with a nilpotent maximal ideal m. Let k denote its residue class field. We claim that $\pi(\operatorname{Spec} A) \cong \pi(\operatorname{Spec} k) \cong \widehat{\mathbb{Z}}$.

The ring homomorphism $A \to k$ induces a continuous group homomorphism $\pi(\operatorname{Spec} k) \to \pi(\operatorname{Spec} A)$. If B is an A-algebra for which $\operatorname{Spec} B \otimes_A k$ is connected, then $\operatorname{Spec} B$ is connected. It follows that the map $\pi(\operatorname{Spec} k) \to \pi(\operatorname{Spec} A)$ is surjective.

Now let $\operatorname{Spec} \ell$ be a connected object $FEt/\operatorname{Spec} k$. Then $\ell \cong k[X]/fk[X]$ for some separable irreducible $f \in k[X]$. Choose $g \in A[X]$ with $(g \mod m[X]) = f$, and such that the leading coefficient of g is a unit. Then $B = A[X]/gA[X]$ is free as an A-module, and $\operatorname{Spec} B \to A$ is unramified. Hence $\operatorname{Spec} B$ belongs to $FEt/\operatorname{Spec} A$, and $B \otimes_A k \cong \ell$. It follows that $\pi(\operatorname{Spec} k) \to \pi(\operatorname{Spec} A)$ is injective. Thus $\pi(\operatorname{Spec} A) \cong \pi(\operatorname{Spec} k)$, and $\pi(\operatorname{Spec} k) \cong \widehat{\mathbb{Z}}$ because it's a finite field.