Some Functors on the Category of Sheaves

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Abstract

In this talk, we'd like to study some functors on categories of sheaves that are induced by continuous maps. More precisely, we will see that a map $\pi : Y \to X$ of schemes will induce several functors between $\mathbf{Sh}(Y_{\text{ét}})$ and $\mathbf{Sh}(X_{\text{ét}})$. We'll be studying properties of these functors and relationships among them. Most of the results of this talk can be generalized to the context of continuous maps between sites. Everything in this talk comes from §I.8 of [1]

1 Background on the Category of Sheaves

Recall that $X_{\text{ét}}$ is the full subcategory of $\mathbf{Sch}/X$ consisting of étale morphisms $U \to X$. The category $X_{\text{ét}}$ comes equipped a Grothendieck topology, the étale topology. We can then form the category $\mathbf{Pre}(X_{\text{ét}})$ of presheaves of abelian groups, together with the reflective subcategory $\mathbf{Sh}(X_{\text{ét}})$ consisting of sheaves of abelian groups. (A subcategory is reflective if the inclusion functor admits a left adjoint.) Finally, recall that the category of (pre)sheaves on a site with values in an abelian category is again an abelian category. This is essentially due to the fact that short exact sequences can be computed stalkwise.

2 Direct Images

Since the pullback of an étale morphism is always an étale morphism, we can push presheaves through morphisms. More precisely, we have the following definition.

Definition 2.1. Let $\pi : Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a presheaf on $Y_{\text{ét}}$. For any object $U \to X$ of $X_{\text{ét}}$, we can define $\pi_*\mathcal{F}(U) = \mathcal{F}(U \times_X Y)$. For any morphism $U \to V$ in $X_{\text{ét}}$, we get a morphism $U \times_X Y \to V \times_X Y$, which induces a morphism $\pi_*\mathcal{F}(V) \to \pi_*\mathcal{F}(U)$. This data determines a presheaf $\pi_*\mathcal{F}$ on $X_{\text{ét}}$ called the *direct image* of $\mathcal{F}$ along $\pi : Y \to X$.

Proposition 2.2. *If $\mathcal{F}$ is a sheaf, then so is $\pi_*\mathcal{F}$.*

The direct image construction gives us a functor $\pi_* : \mathbf{Pre}(Y_{\text{ét}}) \to \mathbf{Pre}(X_{\text{ét}})$. This functor is evidently exact, since exactness is checked at the level of open sets. The fact that $\mathbf{Sh}(Y_{\text{ét}})$ is a reflective subcategory implies the following proposition.

Proposition 2.3. *The functor $\pi_* : \mathbf{Sh}(Y_{\text{ét}}) \to \mathbf{Sh}(X_{\text{ét}})$ is left exact.*

Example 2.4 (Failure of Right Exactness). Consider the a k-scheme $\alpha : X \to \operatorname{Spec} k$ for some algebraically closed field k. The functor $\mathbf{Sh}((\operatorname{Spec} k)_{\text{ét}}) \to \mathbf{Ab}$ sending $\mathcal{F}$ to $\mathcal{F}(\operatorname{Spec} k)$ is an equivalence of categories, and under this equivalence, the direct image functor $\alpha_* : \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}((\operatorname{Spec} k)_{\text{ét}})$ is naturally isomorphic to the global sections functor, which is not generally right exact.

Example 2.5 (Skyscraper Sheaf). Let $i : \overline{x} \to X$ be a geometric point. We again have that $\mathbf{Sh}(\overline{x}_{\text{ét}})$ is equivalent to the category of abelian groups. Under this equivalence, the direct image i_*G of an abelian group G is then the skyscraper sheaf $G^{\overline{x}}$ at $\overline{x}$.

If we have a geometric point $i : \overline{y} \to Y$ and a morphism $\pi : Y \to X$, then we get a geometric point $\overline{x} = \pi(\overline{y}) = \pi \circ i : \overline{y} \to X$. If we have a sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, then the universal property of colimits induces a canonical map

$$(\pi_* \mathcal{F})_{\overline{x}} \to \mathcal{F}_{\overline{y}}.$$

In general, this map is neither injective nor surjective. The following proposition sheds some light on this situation.

Proposition 2.6. *(a) Let $\pi : V \to X$ be an open immersion and let $\mathcal{F}$ be a sheaf on $V_{\acute{e}t}$. Then*

$$(\pi_* \mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in V \\ ? & \text{if } x \notin V. \end{cases}$$

(b) Let $\pi : Z \to X$ be a closed immersion and let $\mathcal{F}$ be a sheaf on $Z_{\acute{e}t}$. Then

$$(\pi_* \mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in Z \\ 0 & \text{if } x \notin Z. \end{cases}$$

(c) Let $\pi : Y \to X$ be a finite map and let $\mathcal{F}$ be a sheaf on $Y_{\acute{e}t}$. Then

$$(\pi_* \mathcal{F})_{\overline{x}} = \bigoplus_{y \mapsto x} \mathcal{F}_{\overline{y}}^{d(y)}$$

where $d(y)$ is the separable degree of $\kappa(y)$ over $\kappa(x)$. In particular, if π is an étale map of degree d between varieties over an algebraically closed field, then

$$(\pi_* \mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}^d.$$

Corollary 2.7. *If $\pi : Y \to X$ is finite or a closed immersion, then the functor $\pi_* : \mathbf{Sh}(Y_{\acute{e}t}) \to \mathbf{Sh}(X_{\acute{e}t})$ is exact.*

The following examples are a theme and variations. The first comes from topology, where sheaves can be pushed forward along continuous maps as one would imagine.

Example 2.8. Let $X \subset \mathbb{R}^2$ be the open disk of radius 1 centered at the origin, let $U = X - \{(0,0)\}$, and let $i : U \to X$ be the inclusion. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U, u)$-module F, where $u \in U$ is some basepoint. (Recall that $\pi_1(U, u) \cong \mathbb{Z}$, so a $\pi_1(U, u)$-module is the same thing as a $\mathbb{Z}[t, t^{-1}]$-module.) Then

$$(i_* \mathcal{F})_{(0,0)} = F^{\pi_1(U,u)},$$

the elements of F fixed by $\pi_1(U, u)$.

Example 2.9. Let $X = \mathbb{A}^1_k$ for some algebraically closed field k, let $U = X - \{0\}$, and let $i : U \to X$ be the open immersion. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U, \overline{u})$-module F, where $\overline{u} \to U$ is some geometric point. (Recall that $\pi_1(U, \overline{u}) \cong \widehat{\mathbb{Z}}$.) Then

$$(i_* \mathcal{F})_0 = F^{\pi_1(U,\overline{u})},$$

the elements of F fixed by $\pi_1(U, u)$.

Example 2.10. Let $X = \operatorname{Spec} R$ for some Henselian discrete valuation ring R, let $U = \operatorname{Spec} K$ where K is the field of fractions of R, and let $i : U \to X$ be the inclusion. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U, \overline{u})$-module F, where $\overline{u} \to U$ is the geometric point corresponding to $K \hookrightarrow K^{\text{sep}}$. (Recall that $\pi_1(U, \overline{u}) \cong \operatorname{Gal}(K^{\text{sep}}/K)$.) Let $I \subseteq \pi_1(U, \overline{u})$ be the subgroup of elements acting trivially on the residue field of R. (That is, I is the *inertia group*.) Then

$$(i_* \mathcal{F})_{\overline{x}} = F^I,$$

the elements of F fixed by $\pi_1(U, u)$, where $\overline{x} \to X$ is the unique closed point.

We record here a result about "functoriality" of the direct image.

Proposition 2.11. *If we have morphisms $\phi : Z \to Y$ and $\pi : Y \to X$ of schemes, then we have $(\pi \circ \phi)_* = \pi_* \circ \phi_*$.*

3 Inverse Image

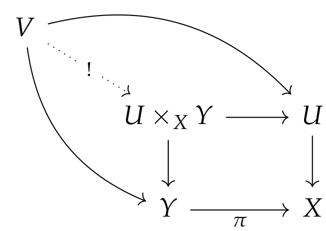
Let $\pi : Y \to X$ be a morphism of schemes and let $\mathcal{F}$ be a presheaf on $X_{\text{ét}}$. We can define a presheaf $\mathcal{F}'$ on $Y_{\text{ét}}$, called the *inverse image* of $\mathcal{F}$, by

$$\mathcal{F}'(V) = \varinjlim \mathcal{F}(U),$$

where the colimit is taken over the collection of commutative diagrams

$$\begin{CD} V @>>> U \\ @VVV @VVV \\ Y @>>{\pi}> X \end{CD} \tag{1}$$

with $U \to X$ étale. Constructing the inverse image presheaf is functorial, yielding a functor $(-)' : \mathbf{Pre}(X_{\text{ét}}) \to \mathbf{Pre}(Y_{\text{ét}})$. If we fill in the above commutative square with the appropriate pullback diagram, we get



If we have a presheaf $\mathcal{G}$ on $Y_{\text{ét}}$, then this second diagram implies that we have natural bijective correspondences between the following collections:

- morphisms $\mathcal{F}' \to \mathcal{G}$,
- morphisms $\mathcal{F}(U) \to \mathcal{G}(V)$ for every commutative diagram 1 for all diagrams compatible in a suitable way, and
- morphisms $\mathcal{F} \to \pi_*\mathcal{G}$.

In other words, we have an adjunction $(-)' : \mathbf{Pre}(X_{\text{ét}}) \rightleftharpoons \mathbf{Pre}(Y_{\text{ét}}) : \pi_*$.

If we take $\mathcal{F}$ to be a sheaf on $X_{\text{ét}}$, then it is not generally true that $\mathcal{F}'$ will be a sheaf on $Y_{\text{ét}}$. To remedy this, we can sheafify in order to get a sheaf $\pi^*\mathcal{F}$ on $Y_{\text{ét}}$. This yields a functor $\pi^* : \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(Y_{\text{ét}})$. For every sheaf $\mathcal{G}$ on $Y_{\text{ét}}$, we have a sequence of natural bijections

$$\mathrm{Hom}_{\mathbf{Sh}(Y_{\text{ét}})}(\pi^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Pre}(Y_{\text{ét}})}(\mathcal{F}', \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Pre}(Y_{\text{ét}})}(\mathcal{F}, \pi_*\mathcal{G}) = \mathrm{Hom}_{\mathbf{Sh}(Y_{\text{ét}})}(\mathcal{F}, \pi_*\mathcal{G})$$

Thus, the adjunction above then gets upgraded to an adjunction $\pi^* : \mathbf{Sh}(X_{\text{ét}}) \rightleftharpoons \mathbf{Sh}(Y_{\text{ét}}) : \pi_*$.

Proposition 3.1. *Let $\mathcal{F}$ be a sheaf on $X_{ét}$, and let $\pi : U \to X$ be an étale morphism. Then we have a natural isomorphism $\mathcal{F}|U_{ét} \cong \pi^*\mathcal{F}$.*

Proposition 3.2. *Let $i : \overline{x} \to X$ be a geometric point and let $\mathcal{F}$ be a sheaf on $X_{ét}$. Then $(i^*\mathcal{F})(\overline{x}) = \mathcal{F}_{\overline{x}}$.*

Proposition 3.3. *If we have morphisms $\phi : Z \to Y$ and $\pi : Y \to X$ of schemes, then we have $(\pi \circ \phi)^* = \phi^* \circ \pi^*$.*

Corollary 3.4. *The functor $\pi^* : \mathbf{Sh}(X_{ét}) \to \mathbf{Sh}(Y_{ét})$ is exact.*

4 Extension by Zero

Again, consider a morphism $\pi : Y \to X$ of schemes. Since the functor π^* admits a right adjoint, we expect π^* to be right exact. However, corollary 3.4 tells us that it is also left exact. Perhaps the functor π^* also admits a left adjoint? Yes! We will construct one in the special case that $\pi : Y \to X$ is an open immersion.

Definition 4.1. Let $j : U \to X$ be an open immersion. Let $\mathcal{F}$ be a presheaf on $U_{\text{ét}}$. For every étale map $\phi : V \to X$, define

$$\mathcal{F}_!(V) = \begin{cases} \mathcal{F}(V) & \text{if } \phi(V) \subseteq U \\ 0 & \text{else} \end{cases}$$

This, together with the obvious restriction maps, yields a presheaf $\mathcal{F}_!$ on $X_{\text{ét}}$.

Again, if $\mathcal{F}$ is a sheaf on $U_{\text{ét}}$, there is no gaurantee that $\mathcal{F}_!$ is a sheaf on $X_{\text{ét}}$. Again, this is not so bad, since we can sheafify to obtain a sheaf $j_!\mathcal{F}$ on $X_{\text{ét}}$. This gives us a functor $j_! : \mathbf{Sh}(U_{\text{ét}}) \to \mathbf{Sh}(X_{\text{ét}})$.

Proposition 4.2. *Let $j : U \to X$ be an open immersion. Then we have an adjunction $j_! : \mathbf{Sh}(U_{\acute{e}t}) \rightleftharpoons \mathbf{Sh}(X_{\acute{e}t}) : j^*$.*

References

[1] James S. Milne. Lectures on etale cohomology (v2.10), 2008. Available at www.jmilne.org/math/.