Etale Cohomology Seminar, Spring 2017
Talk #3: Henselian Rings and Henselization
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Reference. Milne, Lectures on Étale Cohomology, Section 1.4.

Key Words. Stalk of Sheaf, Henselian Ring, Henselization, Strict Henselization, local ring for the étale topology.

1 Notation

In the following, we will assume that X is a variety over an algebraically closed field k. We will denote by (R, m, k) a local ring R with maximal ideal m and residue field $R/m \cong k$.

2 Motivation

The local ring for the étale topology plays a similar role that the stalk of a sheaf plays in algebraic geometry. Recall that for a ringed space (X, O_X), the stalk at a point $x \in X$ is defined as the colimit of $O_X(U)$ over open neighborhoods $U \ni x$, i.e.

$$O_{X,x} = \varinjlim_{U \ni x} O_X(U)$$

3 Definitions

The étale neighborhood of a point $x \in X$ is a pair (U, u) and an étale map $U \to X$ taking $u \in U$ to $x \in X$. Recall from a previous lecture that

- The composition of two étale maps is étale.
- Given two étale neighborhoods (U, u) and (V, v) of (x, X), there is at most one étale map $i : U \to V$ taking u to v and making the relevant diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ & \searrow & \downarrow \\ & & X \end{array}$$

So we can form a directed system of all étale neighborhoods

$$(U, u) \leq (V, v) \Longleftrightarrow \text{there exists a map } (U, u) \to (V, v)$$

. Thus, in analogy with the definition of a stalk, we define

Definition 3.1. The local ring at x for the étale topology is defined as the colimit of all O_U for étale neighborhoods (U, u) of x, i.e.

$$O_{X,\overline{x}} = \varinjlim_{(U,u)} O_U$$

Remark. For an étale neighborhood (U, u) of x, given any open subset $U' \subset U$ containing u, (U', u) is also an étale neighborhood of x. Since we take the colimit over all such neighborhoods, it is really only the stalk at u that matters. So we can take the definition of $O_{X,\overline{x}}$ to be

$$O_{X,\overline{x}} = \varinjlim_{(U,u)} O_{U,u}.$$

4 Properties of $O_{X,\overline{x}}$

Proposition 1. The ring $O_{X,\overline{x}}$ is local, Noetherian and has Krull dimension $\dim X$.

Furthermore, $O_{X,\overline{x}}$ satisfies the conclusion of Hensel's lemma.

Proposition 2. The ring $O_{X,\overline{x}}$ is Henselian (Defined below).

Definition 4.1. A local ring (R, m, k) is Henselian if every monic factorization of a monic polynomial in $R[x]$ into relatively prime factors in the residue field k lifts to a factorization in the ring R. (R, m, k) is called strictly Henselian if it is Henselian and it's residue field is separably closed.

The property of being Henselian is equivalent to the following property: Given a local ring (R, m, k) and $f_1, \cdots, f_n \in R[x_1, \cdots, x_n]$, every common root of their reductions $\overline{f}_1, \cdots, \overline{f}_n$ in k^n at which $\mathrm{Jac}(\overline{f}_1, \cdots, \overline{f}_n) \neq 0$ lifts to a common root of $f_1, \cdots, f_n$ in R^n. This is essentially proven by reinterpreting the problem of lifting a factorization in terms of solving a system of polynomial equations in the coefficients of the factorization.

Definition 4.2. The Henselization of a local ring A is a local ring A^h with a map of local rings $i : A \to A^h$ which satisfies the universal property that any map of local rings from A to another Henselian local ring B factors through i.

Example 1. Consider the ring $R = \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the prime ideal (p). The Henselization of R is the integral closure of R in $\mathbb{Z}_p$ the p-adic integers (I.e. all elements of $\mathbb{Z}_p$ that satisfy a monic polynomial in $\mathbb{Z}_(p)$.

Example 2. The stalk of k-affine n-space at the origin is the local ring $k[x_1, \cdots, x_n]_{(x_1,\cdots,x_n)}$. It's Henselization is the intersection

$$k[[x_1, \cdots, x_n]] \cap k(x_1, \cdots, x_n)^{\mathrm{al}}.$$

This is all those elements of the power series ring in n variables over k that satisfy a polynomial with coefficients in $k(x_1, \cdots, x_n)$.

In both the examples above, we see the Henselization is closely related to the completion of (R, m, k) with respect to its m-adic topology. The Henselization, however, is a much smaller ring and does not include any of the transcendental baggage that completions are usually saddled with. This reflects the fact that the completion of (R, m, k) deals with all Cauchy sequences with respect to the m-adic topology whereas a Henselian ring simply needs to be able to lift *polynomial* factorizations.

Lemma 3. If a map $f : X \to Y$ is étale at $y \in Y$, then it induces an isomorphism of the local rings at $O_{\overline{f(y)},X}$ and $O_{\overline{y},Y}$.

Lemma 4. For any non-singular point $P \in X$, there is a a regular map $\varphi : U \to \mathbb{A}^n$ where U is a Zariski neighborhood of P such that φ is étale at P (and maps P to the origin).

The above lemmas show that the local ring for the étale topology at a non-singular point P of a variety X only depends on $\dim X = n$ since all of them are isomorphic to the Henselization of $k[x_1 \cdots x_n]_{(x_1,\cdots,x_n)}$. which we computed in the example above.

A more intrinsic construction of $O_{X,\overline{x}}$ is the following

Proposition 5. The ring $O_{X,\overline{x}}$ is the Henselization of the stalk $O_{X,x}$.

However, for more general schemes, we have to be more careful. Now let X be an arbitrary scheme.

Definition 4.3. A geometric point of X is a map

$$\overline{x} : \mathrm{Spec}\Omega \to X$$

where Ω is a separably closed field. We can analogously define the local ring $O_{X,\overline{x}}$ and we have the following result

Proposition 6. The ring $O_{X,\overline{x}}$ is the strict Henselization of $O_{X,x}$.

Notice that if X is a variety over $k = \overline{k}$, this agrees with Proposition 5.