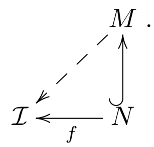
Let T be a topological space, and let $Open(T)$ the small site of open sets of T with inclusions as morphisms. Recall that classically, we define sheaf cohomology by defining an abelian category of sheaves valued in abelian groups, $Sh(T, Ab)$. We have the usual notion of an injective sheaf: it's just the dual notion to a projective object in this abelian category. Spelling this out, an injective sheaf is a sheaf $\mathcal{I}$ s.t. given a monomorphism $N \hookrightarrow M$ and a map $f : N \to \mathcal{I}$, we can extend f to a map $M \to \mathcal{I}$.



Once one shows that this category has *enough injectives* we can talk about an *injective resolution* of an arbitrary object, and define the cohomology of a functor as the right derived functors.

Lemma 1. *(Stacks project 01DL)*
The category of abelian sheaves on a site has enough injectives

What are right derived functors? Well, given an object A in an abelian category $\mathcal{A}$ with enough injectives and a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, to an abelian category $\mathcal{B}$, we take an injective resolution

$$A \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

then define

$$(R^i\mathcal{F})(A) = H^i(0 \to \mathcal{F}(\mathcal{I}_0) \to \mathcal{F}(\mathcal{I}_1) \to \cdots).$$

Sheaf cohomology is defined to be the right derived functors of the global sections functor. Note that I didn't specify that my functor was left exact: this isn't necessary for the definition, but it is necessary to ensure that that zeroth right derived functor is isomorphic to our original functor. This definition plus the horseshoe lemma imply that given a short exact sequence of sheaves

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

we get a long exact sequence

$$0 \longrightarrow H^0_{et}(X, F_1) \longrightarrow H^0_{et}(X, F_2) \longrightarrow H^0_{et}(X, F_2) \longrightarrow H^1_{et}(X, F_1) \longrightarrow \cdots$$

We also see straight from the definition that injective objects have trivial cohomology in positive degee.

However, these definitions are somewhat unnatural, because we're awkwardly avoiding working with the more homotopy-theoretic notion of the derived category of an abelian category. We'll define the derived category now (note, there are several variants):

Given an abelian category $\mathcal{A}$, we form the derived category as follows:

1. Start by forming the category $Ch(\mathcal{A})$ of cochain complexes and chain maps in $\mathcal{A}$.

2. Pass to the homotopy category of cochain complexes: objects are the same, but morphisms are chain homotopy classes of morphisms.

3. Pass to the derived category by localizing at the set of quasi-isomorphisms. Morphisms in the derived category are roofs $X \leftarrow X' \to Y$ where $X' \to X$ is a quasi-isomorphism and $X' \to Y$ is any morphism.

Remark 2. We'll restrict instead to *non-negatively graded* cochain complexes, because our goal at the end of the day is to compute some cohomology.

There are a few different ways to view this construction: One way to say it is that there's a model structure on the category of chain complexes with weak equivalences the quasi isomorphisms and cofibrations the maps which are monomorphisms in all positive degrees, and the derived category is the homotopy category of this model category. Another way to say it is that there's a presentable $(\infty, 1)$-category of chain complexes (the simplicial nerve of the subcategory of cofibrant-fibrant objects of this model category), and this is the associated homotopy category.

Now, how do derived functors relate to the derived category? For the homotopy theorists in the room, we'll know that Quillen functors on a model category induce derived functors on the homotopy category. How do we compute them?

In categories with functorial fibrant/cofibrant replacement functors, we compute the right derived functor $\mathbb{L}(\mathcal{F})(A)$ of a Quillen functor $\mathcal{F}$ by taking a cofibrant replacement of A and applying $\mathcal{F}$ to this cofibrant replacement. Similarly we compute right derived functors by taking a fibrant replacement. If $A \in \mathcal{A}$ is just an element of our abelian category, we regard it as a cochain complex concentrated in degree zero. What does it mean for a chain complex to be fibrant in our above model structure?

Lemma 3. *A quasi isomorphism of non-negatively graded chain complexes $f : C_\bullet \to D_\bullet$ which is a monomorphism in positive degrees is a monomorphism in degree zero.*

Proof. Consider the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker d_0^C & \longrightarrow & C_0 & \longrightarrow & \operatorname{im} d_0^C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_0^D & \longrightarrow & D_0 & \longrightarrow & \operatorname{im} d_0^D & \longrightarrow & 0 \end{array}.$$

Note the the first vertical map is an isomorphism since this is the induced map on H^0, and the right vertical map is a monomorphism since it's a composition of monomorphisms. By, for example, the snake lemma, the fibers of these maps give us an exact sequence

$$0 \to 0 \to \ker f_0 \to 0$$

so that $\ker f_0 = 0$, and we're done. □

Proposition 4. *Say we have a degree-wise complex of injectives $\mathcal{I}_\bullet = 0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$. Then the map $\mathcal{I}_\bullet \to 0$ is a fibration.*

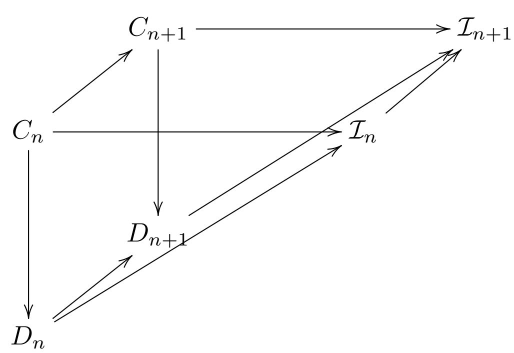
We'll check that it has the RLP w.r.t. acyclic cofibrations. Given a diagram

$$\begin{array}{ccc} C_\bullet & \longrightarrow & \mathcal{I}_\bullet \\ \downarrow & & \downarrow \\ D_\bullet & \longrightarrow & 0 \end{array}$$

where $C_\bullet \to D_\bullet$ is a level-wise monomorphism and a quasi-isomorphism, we'll inductively construct a lift as follows: Since $\mathcal{I}_0$ is injective, there's a lift in the diagram

$$\begin{array}{ccc} C_0 & \longrightarrow & \mathcal{I}_0 \\ \downarrow & \nearrow & \\ D_0 & & \end{array}$$

Now assume inductively that we've defined a map $D_n \to \mathcal{I}_n$. Then we need to define a map $D_{n+1} \to I_{n+1}$ such that



commutes. Making the back triangle commute is trivial, the top rectangle already commutes, as does the side rectangle. Thus the only piece we need to worry about is the diagonal rectangle. For an arbitrary morphism $i_{n+1} : D_{n+1} \to \mathcal{I}_{n+1}$ making the back triangle commute, it's probably not true that $i_{n+1}d_n^D = d_n^{\mathcal{I}} i_n$. However, if we let f_n denote the map $C_n \to D_n$, then by virtue of commutativity of the rest of the diagram, it is true that $i_{n+1}d_n^D f_n = d_n^{\mathcal{I}} i_n f_n$. Thus $(i_{n+1}d_n^D - d_n^{\mathcal{I}} i_n) \circ f_n = 0$. Hence there's an induced map $\phi_n : \mathrm{cok}(f_n) \to \mathcal{I}_{n+1}$. Let's study $\mathrm{cok}(f_n)$. Recall that an exact sequence of chain complexes is by definition a degree-wise exact sequence. Thus the sequence $0 \to C_\bullet \to D_\bullet \to \mathrm{cok}(f)_\bullet \to 0$ is exact, so since f is a quasi-isomorphism, the long exact sequence on homology tells us that $\mathrm{cok}(f)_\bullet$ is exact.

Say $a \in \mathrm{cok}(f)_\bullet$ s.t. $a = d_{n-1}^{\mathrm{cok}(f)}(c)$ for some $c \in \mathrm{cok}(f)_{n-1}$. Then $\phi_n(a) = d_n^{\mathcal{I}} i_n(a)$ (the other term obviously vanishes), where we abuse notation and let a denote a lift of a to D_n. By induction (the diagonal rectangle above commutes if we decrease all the indices by one), $i_n(a) = d_{n-1}^{\mathcal{I}}(t)$ for some $t \in \mathcal{I}_{n-1}$, and hence $\phi_n(a) = 0$. Thus ϕ_n induces a map $\tilde{\phi} : \mathrm{cok}(f_n)/\ker(d_n^{\mathrm{cok}(f)}) \to \mathcal{I}_{n+1}$, (we use exactness of the complex $\mathrm{cok}(f)_\bullet$). But, again by exactness, $\mathrm{cok}(f_n)/\ker(d_n^{\mathrm{cok}(f)}) \cong \mathrm{im}(d_n^{\mathrm{cok}(f)}) \subset \mathrm{cok}(f_{n+1})$ and we get an induced map $\psi : \mathrm{im}(d_n^{\mathrm{cok}(f)}) \to \mathcal{I}_{n+1}$. Because $\mathcal{I}_{n+1}$ is injective, we can extend ψ along the inclusion $\mathrm{im}(d_n^{\mathrm{cok}(f)}) \hookrightarrow \mathrm{cok}(f_{n+1})$ to get a map $\widetilde{\psi} : \mathrm{cok}(f)_{n+1} \to \mathcal{I}_{n+1}$ making the obvious triangle commute. Let ϵ_{n+1} denote the canonical map $D_{n+1} \to \mathrm{cok}(f_{n+1})$. Define $i_{n+1}^* = i_{n+1} - \widetilde{\psi}\epsilon_{n+1}$.

Now, since we're adding something in the cokernel of f_{n+1}, we haven't change the commutativity of the back triangle. Now, let's calculate: $i_{n+1}^* \circ d_n^D(a) = (i_{n+1}(d_n^D(a)) - \widetilde{\psi}\epsilon_{n+1}(d_n^D(a)))$. Since $\epsilon_{n+1}(d_n^D(a)) \in \mathrm{im}(d_n^{\mathrm{cok}(f)})$, the second summand is $\psi(d_n^{\mathrm{cok}(f)}(a))$, which in turn is $\phi_n(a) = i_{n+1}d_n^D - d_n^{\mathcal{I}} i_n(a)$. Thus $(i_{n+1}(d_n^D(a)) - i_{n+1}d_n^D + d_n^{\mathcal{I}} i_n(a) = d^{\mathcal{I}} i_n(a)$, just as desired so that the diagonal rectangle now commutes. By induction we're done.

In summary, to compute the *total right derived functor* of the global sections functor $(R\mathcal{F})(A)$ we can regard A as a cochain complex concentrated in degree zero, then find a quasi-isomorphic co-chain complex of injective objects and evaluate $\mathcal{F}$ level-wise on this complex to get a new complex. Thus the relationship between our two formulas is given by $(R^i\mathcal{F})(A) = H^i((R\mathcal{F})(A))$.

Examples of injective sheaves

1. Fix a scheme X and consider the small étale site $X_{ét}$. If we fix an injective abelian group A and consider the constant sheaf on A, is it injective as a sheaf? Certainly not. Let X be the $\mathbb{C}$-scheme $\mathbb{C}P^n$. Let $A = \mathbb{Q}_p$, which is divisible since it's a field of characteristic zero. Then by the Artin comparison theorem, the étale cohomology of X is the singular cohomology $H^{sing}(X(\mathbb{C}); \mathbb{Q}_p) \cong \mathbb{Q}_p[x]/x^n$ with $|x| = 2$. If the constant sheaf $\mathbb{Q}_p$ was injective, this would have trivial cohomology outside dimension zero.

   Note that this example implies that Zariski-flasque sheaves are NOT acyclic with regards to étale cohomology.

2. Let $\overline{x} = \mathrm{Spec}\, L$, where L is a separably closed field. As Brian mentioned in his talk, the category of abelian sheaves on the étale site of L is equivalent as an abelian category to the category of abelian groups (there are no nontrivial algebraic extensions, and transcendental extensions don't have finite type). It follows that A is injective when considered as a constant sheaf on $Sh(\overline{x}_{ét})$. Now if we have a geometric point $i : \overline{x} \to X$, the pushforward $i_*(A)$ is an injective sheaf on X. We saw in Brian's talk that $i_*(A)$ is the skyscraper sheaf $A^{\overline{x}}$ at $\overline{x}$.

3. If we have an injective sheaf in the étale topology, is injective when viewed as a Zariski sheaf? Recall that we have a morphism of sites $\tilde{f} : X_{ét} \to X_{Zar}$ which corresponds to a map of categories $f : X_{Zar} \to X_{ét}$ that sends a Zariski map to itself. We get induced maps $f_* : Sh(X_{ét}) \to Sh(X_{Zar})$ with $(f_*F)(U) = F(f(U))$ and f^* the left adjoint. It's still true that f^* is exact so that f_* preserves injectives.

Eilenberg-Steenrod Axioms

Dimension

Let $x = \mathrm{Spec}\, k$ for a field k and $\overline{x} = \mathrm{Spec}\, k^{sep}$ for some separable closure of k. Ningchuan showed that there's an equivalence from the category of sheaves on x_{et} to the category of discrete G-modules where

$G = Gal(k^{sep}/k)$ gotten by sending a sheaf to the stalk over the geometric point $\overline{x} \to x$. It follows that $H^r(x, \mathcal{F}) \cong H^r(G, \mathcal{F}_{\overline{x}})$, so that if x is a geometric point, we have cohomological vanishing in positive dimensions.

Note that in the Zariski topology we DON'T need a geometric point to get the dimension axiom.

Exactness

Let Z be a closed subscheme of X, and let $U = X - Z$. For any sheaf $\mathcal{F}$ on X_{et}, define

$$\Gamma_Z(X, \mathcal{F}) = \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})),$$

the group of sections of $\mathcal{F}$ with support on Z. The functor $\mathcal{F} \to \Gamma_Z(X, \mathcal{F})$ is left exact, and we denote its rth right derived functor by $H^r_Z(X, -)$.

Theorem 1. *For any sheaf $\mathcal{F}$ on X_{et} and closed $Z \subset X$, there's a LES*

$$\cdots \longrightarrow H^r_Z(X, \mathcal{F}) \longrightarrow H^r(U, \mathcal{F}) \longrightarrow H^{r+1}_Z(X, \mathcal{F})$$

which is functorial in the pairs $(X, X - Z)$ and $\mathcal{F}$.

The proof reduces this to a proof about a LES of Ext groups associated to the exact sequence

$$0 \to j_! j^* \mathbb{Z} \to \mathbb{Z} \to i_* i^* \mathbb{Z} \to 0$$

where $j : U \to X$ is the open immersion above, and i is the closed embedding $Z \to X$.

Excision

Recall that in the topological setting, one way to phrase the excision isomorphism is that a map of pairs $(X - Z, A - Z) \to (X, A)$ induces an isomorphism on cohomology. In particular, if $V = X - T$ is an open nbd of a closed subset Z of X (so that $T \subset Z^C$, then the map $(X - T, X - Z - T) = (V, V - Z) \to (X, X - Z)$ induces an isomorphism on cohomology. Recall that the de Rham cohomology of the pair $(X, X - Z)$ is essentially "forms supported on Z" (we need to worry because in de Rham cohomology, the subspace is closed, not open).

Theorem 2. *Let $\pi : X' \to X$ be an étale map and let $Z' \subset X'$ be a closed subscheme of X' s.t.*

1. *$Z = \pi(Z')$ is closed in X, and the restriction of π to Z' is an isomorphism of Z' onto Z, and*
2. *$\pi(X' - Z') \subset X - Z$.*

Then, for any sheaf $\mathcal{F}$ on X_{et}, the canonical map $H^r_Z(X_{et}, \mathcal{F}) \to H^r_{Z'}(X'_{et}, \mathcal{F}\mid_{X'})$ is an isomorphism for all r.

Probably the most helpful way to think about this theorem is to let $X' \to X$ be an open immersion $U \to X$ with Z' a closed subscheme of U.

Homotopy

Classically, this says that homotopic maps induce the same map on cohomology. It's proved by proving that homotopic maps induce chain homotopies of the singular complex.

Definition 5. Two codimension one subvarieties Z_1, Z_2 of X are rationally equivalent if they're part of the family parametrized by a nice subvariety of $\mathbb{X} \times P^1$.

Theorem 3. *Two morphisms ϕ, ϕ' define the same map on étale cohomology if their graphs are rationally equivalent.*