SITES AND SHEAVES

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ABSTRACT. This talk is given at the étale cohomology reading seminar in 2017. It covers §5 and §6 in [3].

1. SITES

1.1. **Grothendieck topology.** Let X be a topological space. Recall that a presheaf $\mathscr{F}$ on X is a functor

$$\mathscr{F} : \mathrm{Open}(X)^{op} \to \mathrm{Set},$$

where Open(X) is the category of open subsets of X with inclusions. Here Set can be replaced by Ab or $R-$mod, etc. $\mathscr{F}$ is a sheaf, if for any open covering $\{U_i \mid i \in I\}$ of an open subset $U \subset X$, we have the following equalizer sequence:

$$\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i\neq j\in I} \mathscr{F}(U_i \cap U_j).$$

Now we can generalize this definition of sheaves on X to sheaves on a category $\mathcal{C}$. To do that, we first need the notion of covering in a category.

Definition 1.1. A **Grothendieck topology** J on a category $\mathcal{C}$ with finite limits is a collection of sets of morphisms $\{U_i \to U\}$ ("coverings"), satisfying

(1) Isomorphisms are coverings,
(2) $\{U_i \to U\} \in J$ and $\{V_{ij} \to U_i\} \in J$ implies $\{V_{ij} \to U\} \in J$,
(3) For any morphism $V \to U$, $\{U_i \to U\} \in J$ implies $\{V \times_U U_i \to V\} \in J$.

A category (with finite limits) with a Grothendieck topology is called a **site**.

Clearly, the usual notion of covering defines a Grothendieck topology on Open(X). (Here fiber products are intersections of open subsets.) Having defined coverings in a category, it's natural to define

Definition 1.2. A **sheaf** on a site $(\mathcal{C}, J)$ is a functor $\mathscr{F} : \mathcal{C}^{op} \to \mathrm{Set}$, such that for any covering $\{U_i \to U\} \in J$,

$$\mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \prod_{i\neq j} \mathscr{F}(U_i \times_U U_j), \tag{1}$$

is an equalizer sequence.

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1.2. **Examples of sites.**

Examples 1.3. The obvious example is Open(X) with the usual notion of coverings. Similarly, the category of topological spaces together with usual notion of coverings is a site.

There's another Grothendieck topology on the category of spaces. We say a collection of maps $\{U_i \to U\}$ is a covering if the image is an open cover of U and $U_i \to U$ is a covering space of its image. The site is denoted by $(\text{Space})_{ét}$.

Examples 1.4. For schemes (rings), we have the following Grothendieck topologies.

- **Zariski site $X_{\mathbf{Zar}}$:** $(\text{Open}(X), J)$ with the usual notion of coverings in the Zariski topology of X.
- **Big Zariski site:** (X-Sch, J) with J the set of X-morphisms $\{U_i \to U\}$ for which each $U_i \to U$ is an open embedding and the image is an open covering of U.
- **Small étale site $X_{\text{ét}}$:** Let $\acute{E}t(X)$ be the full subcategory of X-Sch whose objects are étale morphisms $U \to X$. A collection of morphisms $\{U_i \to U\}$ is a covering if their image is a covering of U, i.e.

$$\coprod U_i \to U$$

  is surjective.
- **Big étale site $X_{\text{Ét}}$:** Change the category of the small étale site over X to X-Sch and keep the Grothendieck topology, we get the big étale site.
- **Flat site X_{fl}:** Let $\mathcal{C} = X$-Sch. A collection of morphisms $\{U_i \to U\}$ is covering if each morphism $U_i \to U$ is flat and of finite type and the image of the morphisms is a covering.
- **Completely decomposed topology X_{cd}:** Let $\mathcal{C} = \acute{E}t(X)$ or X-Sch. The Nisnevich (completely decomposed) topology is defined by saying a set of morphisms $\{U_i \to U\}$ is a covering if each one of them is étale and for any $x \in U$, there exist an $i \in I$ and an $x_i \in U_i$ a preimage of x such that the induced map on residue fields is an isomorphism.

Remark 1.5. We can translate the flat topology on schemes to a topology on the opposite category of rings, where a map $R \to R'$ is a covering of R if this map is faithfully flat.

Definition 1.6. A map of sites $\mathscr{F} : (\mathcal{C}_1, J_1) \to (\mathcal{C}_2, J_2)$ is a functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ that preserves fiber products and the Grothendieck topology. Such functor is called continuous.

From definition we have obvious continuous maps of sites

$$X_{\text{Fl}} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\text{cd}} \to X_{\text{Zar}}.$$

2. Sheaves on étale sites

2.1. **A useful criterion.**

Before we start to give examples, there's a useful criterion for a presheaf over $X_{\text{ét}}$ to be a sheaf.

Theorem 2.1. *To show a presheaf $\mathscr{F}$ on $X_{ét}$ is a sheaf, it suffices to check (1) on Zariski open coverings and étale covering maps $V \to U$ of affine schemes.*

In the following example, when checking the (1) on an étale covering map of affine schemes, we'll only use the fact this map is faithfully flat.

Example 2.2. The first example is the structure sheaf on $X_{\text{ét}}$. The structure sheaf defined by

$$\Gamma(U, \mathcal{O}_X^{\text{ét}}) = \Gamma(U, \mathcal{O}_U)$$

for any étale map $U \to X$. Clearly, this sheaf satisfies (1) for Zariski open coverings. To check (1) on an étale (faithfully flat) map of affine schemes, we need the following lemma:

Lemma 2.3. *If $A \to B$ is faithfully flat, then there is an exact sequence:*

$$0 \to A \to B \to B \otimes_A B, \tag{2}$$

where the second map is $b \mapsto 1 \otimes b - b \otimes 1$.

Example 2.4. Representable functors are sheaves. For any $Z \in X$-Sch, we define a functor $\mathscr{F}_Z : \text{Ét}(X) \to \text{Set}$ by $U \mapsto \hom_X(U, Z)$. This a presheaf on $X_{\text{ét}}$ and satisfies (1) for Zariski coverings. Lemma 2.3 shows it also satisfies sheaf axiom for étale covering of affine schemes.

If Z is an X-group scheme, then this sheaf is actually valued in groups.

(1) For $Z = \mu_n = X \times \operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$, $\mathscr{F}_Z(U)$ gives the n-th roots unity on $\Gamma(U)$.
(2) For $Z = \mathbb{G}_a = \mathbb{A}^1_X$, $\mathscr{F}_Z$ yields the abelian group of sections.
(3) For $Z = \mathbb{G}_m = X \times \operatorname{Spec} \mathbb{Z}[t^{\pm 1}]$, $\mathscr{F}_Z$ is the sheaf of units.

Example 2.5. Clearly, the locally constant sheaf on X_{Zar} is also a sheaf on $X_{\text{ét}}$.

Example 2.6. A sheaf of coherent $\mathcal{O}_X$-modules induces a sheaf of $\mathcal{O}_X^{\text{ét}}$-modules.

2.2. **Galois coverings and sheaves over a point.** What is a sheaf over a point $\operatorname{Spec} k$ in étale topology. To answer this question, we need to introduce Galois covering.

Definition 2.7. Let $\varphi : Y \to X$ be faithfully flat and G be a finite group acting on Y over X on the right. φ is called a Galois covering if

$$Y \times G \to Y \times_X Y, \quad (y, g) \mapsto (y, y \cdot g)$$

is an isomorphism, where $Y \times G$ means disjoint union of $|G|$ copies of Y.

An A-algebra B is Galois if there is a group G acting on B and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a Galois covering with the G-action.

Example 2.8. Let k be a field and $K = k[t]/(f(t))$, where $f(t)$ is monic irreducible polynomial over k. If

$$f(t) = \prod f_i(t)^{e_i}$$

over K with f_i irreducible over K and $e_i \geq 1$, then by Chinese Remainder Theorem, we have

$$K \otimes_k K \cong \prod K[t]/(f_i(t)^{e_i}).$$

So K is a Galois algebra over k if f splits as a product of distinct linear factors over K, i.e. K is a splitting field of f and if in that case K/k is a Galois extension with Galois group G.

Proposition 2.9. *Let $Y \to X$ be a Galois covering with a right group action by G and let $\mathscr{F}$ be a presheaf on $X_{\acute{e}t}$ that maps disjoint unions to products. Then $\mathscr{F}$ is a sheaf iff $\mathscr{F}(X) \to \mathscr{F}(Y)$ identifies $\mathscr{F}(X)$ with $\mathscr{F}(Y)^G$.*

Now, let's consider sheaves of abelian groups over $\mathrm{Spec}(k)_{\text{ét}}$. By the discussion above, we know they are equivalent to functors from Galois algebras over k to abelian groups such that $\mathscr{F}(\prod A_i) = \oplus \mathscr{F}(A_i)$ and $\mathscr{F}(k') = \mathscr{F}(K)^{\mathrm{Gal}(k'/K)}$ for any finite Galois extensions K/k' with k' finite over k.

Let k^{sep} be a separable closure of k and $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. There's an equivalence of categories between sheaves on $\mathrm{Spec}(k)_{\text{ét}}$ and discrete G-sets given by

$$\mathscr{F} \mapsto M_{\mathscr{F}} = \lim \mathscr{F}(\mathrm{Spec}\, k'),$$

where $k' \subset k^{\mathrm{sep}}$ and is finite Galois over k and conversely

$$M \mapsto (A \mapsto \hom_G(F(A), M)),$$

where $F(A) = \hom_k(A, k^{\mathrm{sep}})$.

2.3. **Stalks and skyscraper sheaves.** For a variety X over an algebraically closed field k, define the stalk of a presheaf on $X_{\text{ét}}$ at $x \in X$ by

$$\mathscr{F}_{\bar{x}} = \operatorname*{colim}_{x \in U \to X \text{ étale}} \mathscr{F}(U).$$

Similarly, we can the stalk of a presheaf over the étale site of a scheme at its geometric point.

Examples 2.10. (1) For the étale structure sheaf, the stalk at x is $\mathscr{O}_{X,\hat{x}}$.
(2) For representable sheaves Z, the stalk is $Z(\mathscr{O}_{X,\hat{x}})$.
(3) For sheaves of modules, $M_{\hat{x}} = M_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,\hat{x}}$.
(4) For the sheaf $\mathscr{F}$over $\mathrm{Spec}(k)_{\text{ét}}$, the stalk at k^{sep} is $M_{\mathscr{F}}$.

Let's define the skyscraper sheaf now. Let $x \in X$ be a geometric point and let A be an abelian group. Define the skyscraper sheaf A^x by setting for an étale map $\varphi : U \to X$, define

$$A^x(U) = \bigoplus_{\hom_X(x,U)} A.$$

We get a sheaf over $X_{\text{ét}}$ with a natural isomorphism $\hom(\mathscr{F}, A^x) \to \hom(\mathscr{F}_{\hat{x}}, A)$.

References

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