Why should we study étale cohomology?

Algebraic topology has been remarkably successful in defining invariants to differentiate between topological spaces. If we have a smooth scheme X over $\operatorname{Spec}\mathbb{C}$, then we have a functor $Sm_{\mathbb{C}} \to Top$ which sends X to the set $X(\mathbb{C})$ of complex points with the analytic topology (just the subspace topology of the normal euclidean topology on $\mathbb{C}^n$ for affine schemes). Thus we can differentiate between schemes by studying these assocaited topological spaces and applying the tools of algebraic topology.

Unfortunately, not all schemes live over $\operatorname{Spec}\mathbb{C}$, so we'd like to find some other way of using our topological toolkit to study schemes. In some sense, étale cohomology is the generalization of this construction to arbitrary schemes.

Why doesn't the Zariski topology work?

There are many quirks to the Zariski topology which make it behave much differently than the topologies we're used to. In essence, Zariski open sets are too "large". Here I'll list some of the differences:
Theorem 1. *Let X be an irreducible topological space. Then $H^i(X; A) = 0$ for $i > 0$ if A is a constant sheaf of abelian groups.*

Proof. Constant sheaves are flasque over irreducible spaces (for an irreducible space it's just the constant presheaf), and flasque sheaves are acyclic (proof of this is longer). □

The key difference here is that many of the schemes we consider in algebraic geometry are in fact irreducible in the Zariski topology, whereas this is incredibly rare for the spaces we consider in algebraic topology.
Remark 1. Maps which induce isomorphisms on tangent spaces at a point are not Zariski local isomorphisms near that point.
Example 2. The power map $\mathbb{C} \to \mathbb{C}$, $x \mapsto x^n$.

Proof. Identifying $\mathbb{C}$ with the tangent space to $\mathbb{A}^1$ at a point, the induced map on tangent spaces at X is multiplication by nX^{n-1}. This is an isomorphism unless $X = 0$. Now, however, the induced map on function fields between any two open subsets is $\mathbb{C}(t) \to \mathbb{C}(t)$, $t \mapsto t^n$, which is not an isomorphism, so that the map can't be locally trivial. In essence the problem is that the usual branch cut to define square root that we make in complex analysis is not a Zariski open subset of $\mathbb{A}^1$ (since these are the cofinite sets). □

Remark 3. Covering spaces (or more general fiber bundles) are not Zariski locally trivial.
Example 4. Consider the curve $X \subset \mathbb{C} \times \mathbb{C}$ defined by $g(x, y) = x^2 + xy + \frac{1}{4}(y^2 + y) = 0$. (The real points of this form an ellipse). Let π denote the projection onto the second factor, so that $\pi((x, y)) = y$. For a fixed y, the given polynomial in x has a double root precisely when $\sqrt{y^2 - y^2 + y} = 0$. In other words, when $y = 0$. There is a single point $(0, 0)$ mapping to 0, so consider $X - \{(0, 0)\} \to \mathbb{C} - \{0\}$.

I claim that this is a topological covering. In fact, take the open sets corresponding to $\mathbb{C} - [0, \infty)$ and $\mathbb{C} - (-\infty, 0]$. Then these are evenly covered neighborhoods with sections given by the quadratic formula, since we've just taken the usual branch cuts which make $y \mapsto \sqrt{y}$ continuous.

However, I claim that this is not a locally trivial covering in scheme land. For this, the setup is a map $\operatorname{Spec} B \to \operatorname{Spec} A$, where $A = \mathbb{C}[y, y^{-1}]$, and $B = A[x]/gA[x]$. Note that if this were a trivial covering, there would be an affine nbd $\operatorname{Spec} S_x$ of every point x such that the map locally had the form $\coprod \operatorname{Spec} S_x \to \operatorname{Spec} S_x$. In particular, for any $x \in X = \operatorname{Spec} B$, the induced map on stalks would be an isomorphism. Now look at the generic point y of $\operatorname{Spec} A$. The stalk at this point is the fraction field $Frac(A)$, and we see that the fiber over y is $\operatorname{Spec}(Frac(A) \otimes_A \operatorname{Spec} B)$. I claim that this is $Frac(B)$. One way to see this is that the generic point of $\operatorname{Spec} B$ maps to the generic point of $\operatorname{Spec} A$ since it's a dominant morphism. Another way is the following: given any $b \in B$, we have a monic irreducible polynomial $f(x) = x^n + \cdots + a_1 x + a_0$ in $A[x]$ s.t. $f(b) = 0$ (this is an integral extension by construction). Then in the usual way we construct b^{-1} as $\frac{-1}{a_0}(b^n + \cdots + a_1 b)$. This is an element of the above tensor product in the obvious way. Thus the canonical map $Frac(A) \otimes_A B \to Frac(B)$ coming from definition of coproduct is an isomorphism.

It follows that the induced map on stalks from the generic point of Spec B to the generic point of Spec A is not an isomorphism, and hence this can't be a Zariski locally trivial map. We will see that it is indeed a finite étale map, and finite etale maps are "locally trivial" in the étale topology.

Applications of étale cohomology

- Via the Quillen-Lichtenbaum conjecture, étale cohomology is the E_2-term of a spectral sequence converging in some range to algebraic K-theory. This conjecture is a consequence of the Beilinson-Lichtenbaum conjecture, which was proved by Voevodsky in 2008. More specifically, the Beilinson-Lichtenbaum conjecture says that for a smooth scheme X over a field k, the motivic cohomology groups $H^{p,q}(X;\mathbb{Z}/n)$ are isomorphic to $H^p_{et}(X;\mu_n^{\otimes q})$ if $p < q$, and this together with convergence of the slice spectral sequence can be shown to imply the Quillen-Lichtenbaum conjecture about étale K-theory agreeing with Quillen K-theory with finite coefficients in a certain range.
- The étale cohomology of a field is just Galois cohomology.

Goals for the course

- Example-centric. If possible, try to come up with completely worked examples that are not in Milne in the course of writing your talk.
- Don't completely shrug off the commutative algebra, at least mention the theorems needed to complete a proof and where to find them.
- Because of limited time, most of the material in this course will have to be focused on foundations rather than hard applications. Encourage group interaction outside of the course to understand these applications.
- Emphasize the connections with algebraic topology.